The Cauchy distribution has well-known, unique features. For example, it does not have any moment; its sample mean has the same Cauchy distribution. Pillai and Meng (2016) give another surprising result: for two independent and identically distributed multivariate normal random vectors with mean zero and any covariance matrix, the (nonnegatively) weighted average of their component-wise ratio follows a standard Cauchy distribution. It suggests that the Cauchy distribution has high tolerance to dependence structures. Liu and Xie (2020) show that, for a standard multivariate Cauchy distribution with any dependence structure (copula), the nonnegatively weighted average of the marginal Cauchy variables has the same tail distribution as that of a standard Cauchy distribution. That is, a linear combination of correlated Cauchy variables with arbitrary dependence strictures has the same tail as that of a Cauchy variable. This property makes the Cauchy distribution an angel in combining p-values in multiple testing.

Consider a global testing problem where we have a collection of null hypotheses. Each individual hypothesis test has its own p-value. Under the global null hypothesis, these p-values are uniformly distributed, but could be dependent. Liu and Xie (2020) proposed a Cauchy combination test, where the test statistic is a linear combination of Cauchy variables obtained by transforming the p-values with the quantile function of standard Cauchy distribution. The global null hypothesis is rejected when the statistic is overly large relative to a standard Cauchy tail. Unlike Fisher’s combination of p-values, this test does not require the p-values to be independent, can handle large number of hypotheses with sparse signals, and allows incorporation of prior information on weighting the individual tests. The test has been picked up quickly in practice such as genetic studies. It has also been adapted to address some of its limitations (Chen, 2022).

References: